# LECTURE 2 ALGEBRAIC SOLUTIONS: JORDAN-PAINLEVE-BOULANGER-RISCH

TWO APPROACHES JPBR & FUCHS

JPBR ALGORITHM:

$k = \bar{\mathbb{Q}}(x)$ $L(y) = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y$

; Assume $L$ irreducible $\Rightarrow$ If $\exists$ algebraic solution $\Rightarrow$ all solns algebraic

- Assume only regular sing pts ; rat'l exponents

Algorithm: 2 steps

STEP 1: Decide if $L(y) = 0$ has a solution s.t. $u = y'/y$ algebraic over $k$

STEP 2: Given $u$ algebraic over $\bar{\mathbb{Q}}(x)$ decide if there is a $y$ algebraic over $\bar{\mathbb{Q}}(x)$ s.t. $y'/y = u$

STEP 1: JORDAN'S THM: There is a function $J(n)$ s.t. if $G$ is a finite subgroup of $GL_n(\mathbb{C})$ then $\exists$ an abelian, normal $H \triangleleft G$ with $|G:H| \leq J(n)$

JORDAN (1878), SCHUR, SPEISER (1945) $n!\,12^{n\pi(n+1)+1}$ $\pi$ = PRIME COUNTING

COLLINS $(n+1)!$ $n \geq 71$ Sharp $S_{n+1} < GL_n$

Non-constructive proof TAO'S BLOG

Need weaker thm: $\exists$ $H$ abelian $H < G$ $|G:H| \leq A(n)$

$A(n) < J(n)$ $A(2) = 12$, $A(3) = 360$, $A(4) = 25920$

Galois Consequence: $K$ PV extension of $k$

Assume $K$ algebraic over $k$ $G = DGal$

- $K^G = k \Rightarrow K$ is a Galois extension $Gal = G$

Assume $H \subset G$ $H$ abelian $\Rightarrow H$ is diagonalizable WRT basis $y_1 \cdots y_n$

So $\sigma y_1 = \xi y_1 \Rightarrow y_1'/y_1$ fixed by $G$

$$\begin{array}{l} K \\ K^H \\ \\ k \end{array} \quad H, \; G \qquad \begin{array}{l}[K:k] = |G| \\ [K:K^H] = |H| \end{array} \Rightarrow [K^H:k] = \text{index of } H \text{ in } G \le A(n) \le J(n)$$

$\therefore \; [k(u):k] \le A(n)$ $\qquad u = \dfrac{y_1'}{y_1}$ $\qquad$ JORDAN

Look for minimal poly of $u$

$$P(u) = b_m u^m + \cdots + b_0 \quad b_i \in \mathbb{Q}[x] \quad m \le A(n)$$

Painlevé - Boulanger:

(i) Can bound degree of $b_m$ in terms of $m$ and exponents

(ii) If degree $b_m \le r \Rightarrow \deg b_i \le r + m - i$

$u$ satisfies RICCATI EQN:

$$R(u) = P_n(u, \ldots, u^{(n-1)}) + a_{n-1} P_{n-1}(u, \ldots, u^{(n-1)}) + \cdots + a_0 = 0$$

$P_0 = 1, P_i = P_{i-1}' + u P_{i-1}$ $\qquad P_0 = 1 \quad P_1 = u \quad P_2 = u' + u^2 \ldots$

$$y'' - ay \Rightarrow \quad \begin{array}{l} u' + u^2 - a = 0 \\ u' = a - u^2 \\ u'' = a' - 2uu' \\ \vdots \end{array}$$

Let coeff of powers of $x$ in $P(u) = b_m u^m + \cdots + b_0$ be indeterminates

"$b u^m + \cdots + b_0 \Rightarrow R(u)=0$" gives polynomial conditions on these coeff.

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To show (ii):

- $u_1 \ldots u_m$ roots of $P(u)=0$
- Each $u_i = \dfrac{z_i'}{z_i}$ order at $\infty$ is at least $-1$

  $L(z_i)=0$

- $b_0/b_m = \pm u_1 \cdots u_m \Rightarrow$ order at $\infty \geq -m$

  $b_1/b_m = \pm \sum u_1 \ldots \hat{u_i} \ldots u_m \Rightarrow$ order at $\infty \geq -(m-1)$

  $\vdots$

  $b_{m-1}/b_m = -\sum u_i \Rightarrow \ldots \geq -1$

order at $\infty$ $b_i/b_m = \deg b_m - \deg b_i \geq -(m-i)$

$r - \deg b_i \geq -(m-i)$

$-\deg b_i \geq -r-(m-i)$

$\deg b_i \leq r+m-i$

To show (i):

$$\frac{b_{m-1}}{b_m} = \pm(u_1 + \cdots + u_m) = \frac{z_1'}{z_1} + \cdots + \frac{z_m'}{z_m} = \frac{(z_1 \cdots z_m)'}{z_1 \cdots z_m}$$

Zeros of $b_m$ = zeroes & poles of $z_i$

& $b_m$ will have a simple zero at such a point

Let $M$ = max number of poles of some $z_i$
$N$ = " " " zeroes " " $z_i$

degree of $b_m \le m(M+N)$

Can bound $M \le$ # of singular points
Need to bound $N$

$$\frac{(z_1 \cdots z_m)'}{z_1 \cdots z_m} = \frac{b_0}{b_m} \in \bar{\mathbb{Q}}(z) \text{ and } z_1 \cdots z_m \text{ algebraic}$$

so

$$z_1 \cdots z_m = f^{1/t} \text{ for some } t \in \mathbb{N}_{>0}$$

$$\Rightarrow z_1 \cdots z_m = \frac{\prod (x-a_i)^{\alpha_i} \prod (x-b_i)^{\beta_i}}{\prod (x-\tilde{a}_i)^{\tilde{\alpha}_i}}$$

$a_i, \tilde{a}_i$ singular pts $\quad \alpha_i, \tilde{\alpha}_i$ - sums of exponents
$b_i$ - zeroes not sing pts $\Rightarrow \underline{\underline{\beta_i \in \mathbb{N}_{>0}}}$

Only finitely many choices for $\alpha_i, \tilde{\alpha}_i$,
FIX ONE
order of each $z_i$ at $\infty$ bounded by $\ell$

order of $z_1 \cdots z_m$ at $\infty$

$$\sum \alpha_i - \sum \tilde{\alpha}_i + \sum \beta_i \le m\ell.$$

$$N \le \sum \beta_i \le m\ell - \sum \alpha_i + \sum \tilde{\alpha}_i \quad \#$$

STEP 2 Given $u$ algebraic over $\overline{\mathbb{Q}}(x)$
decide if $\exists\, y$ alg $/ \overline{\mathbb{Q}}(x)$ s.t. $y'/y = u$

Note: $y$ alg over $\overline{\mathbb{Q}}(x) \Rightarrow y^m \in \overline{\mathbb{Q}}(x,u)$

To decide $\exists ?\ m \in \mathbb{N}_{>0}$ & $z \in \overline{\mathbb{Q}}(x,u)$ s.t. $z'/z = mu$

yes $\Leftrightarrow$ $y = z^{1/m}$ satisfies $y'/y = u$

$\Leftrightarrow \exists\, z, m \quad \frac{dz}{z} = mu\,dx = m\omega \qquad \omega = u\,dx$

Let $X$ be a non-sing model of $\overline{\mathbb{Q}}(x,u)$

Let $n_P = \mathrm{Res}_P(u\,dx) \qquad n_P \in \mathbb{Q}$ (otherwise, NO)

Can assume $n_P \in \mathbb{Z}$ (else replace $\omega$ by $N\omega$)

Let $D = \sum n_P P$

Procedure:

(a) Decide if $\exists\, m$ s.t. $mD = (f)$ for $f \in \overline{\mathbb{Q}}(x,u)$ and if so find smallest $m$.

(b) For this $m, f$ decide if $\frac{f'}{f} = mu$

This is OK:

- if $\exists\, m, z$ s.t. $z'/z = mu \Rightarrow mD = (z)$
- $\{m \mid mD \text{ is principal}\} = (\tilde{m})$ ideal in $\mathbb{Z}$

$\Rightarrow m = l\tilde{m}$

$\tilde{m} D = (\tilde{z}) \Rightarrow z = \tilde{z}^{l}$

$$\Rightarrow \underset{\tilde{m} l u}{\overset{\parallel}{mu}} = \frac{z'}{z} = l\frac{\tilde{z}'}{\tilde{z}} \Rightarrow \tilde{m} u = \frac{\tilde{z}'}{\tilde{z}}$$

So if some $m, z$ exist then minimal $m$ works.

Enough to find smallest $m$ s.t. $mD = (f)$

$X$ defined over number field $k_0$

$\mathrm{Jac}(C) =$ divisors degree $0$ / principal div.

has finite torsion (Mordell-Weil)

Let $\mathfrak{p}_1, \mathfrak{p}_2$ be primes in $k_0$ s.t. $X$ has good reduction mod $\mathfrak{p}_i$ (irred, non-sing)

$X, X_1, X_2$

$J, J_1, J_2$

Good reduction gives a homom

$J \to J_i$

which is an isom on divisors order prime to $p_i$

($\mathfrak{p}_i$ lies above $p_i$)

Assume $D$ has ord $m = p_1^{n_1} p_2^{n_2} M$ $\quad (M, p_1 p_2) = 1$

$D_{\mathfrak{p}_1}$ has order $p_1^{\bar{n}_1} p_2^{n_2} M$ $\quad \bar{n}_1 \le n$

$D_{\mathfrak{p}_2}$ " " $p_1^{n_1} p_2^{\bar{n}_2} M$ $\quad \bar{n}_2 \le n$

Each $J_{\mathfrak{p}_i}$ finite $\Rightarrow$ can bound each of these

$\le (1+\sqrt{q})^{2g}$ Weil $\Rightarrow$ Can bound order of $D$ by $N_D$

Now decide if $D, 2D, \dots N_D D$ is principal

continue

A) To decide if all solutions of $L(y)=0$ are alg

Procede by induction $L = L_2 \circ L_1$ $L_1$ irred

- All solutions alg $\Leftrightarrow$
  - All solutions of $L_1(y)=0$ $L_2(y)=0$ alg.

and

  - $\{y_1 .. y_r\}$ basis of $Soln(L_1)$
    $\{z_1 .. z_s\}$ " " $Soln(L_2)$

    For each $z_i$, $\exists u_{i1}, .., u_{ir} \in C(x)(y_1 .. y_r, z_1 .. z_s)$

    s.t. $L_1(\sum u_{ij} y_j) = z_i$

B) Given $L(y)$ and power series $z = \sum a_i x^i$, $L(z)=0$

Decide if $z$ is algebraic.

The vector space $V$ of alg solns of $L(y)=0$ is D.Gal invariant $\Rightarrow L = L_2 \circ L_1$

$Soln(L_1) = V$

- Can find $L_1$ using irred factorization of $L$ and JPBR algorithm
- Decide if $L_1(z)=0$.

  $L = L_2 \circ L_1(z)$ so

  decide if $L_1(z) = 0 \mod x^N$

  $N >$ largest integer exponent of $L_2$